MATH2050C Selected Solution to Assignment 8

Section 4.1

(9d). We use ε-δ definition. Consider

$$\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| = \left|\frac{2x^2-3x+1}{2(x+1)}\right| = \left|\frac{2x-1}{x+1}\right||x-1|.$$

We make a first choice $\delta_1 = 1/2$. Then for $|x-1| < 1/2$, that is, $1/2 < x < 3/2$. Then $|2x-1|/|x+1| \le 4/3$. Therefore, for $\delta = \min\{\delta_1, 3\varepsilon/4\}$, we have

$$\left|\frac{x^2-x+1}{(x+1)-1/2}\right| < \frac{4}{3}|x-1| < \varepsilon \ ,$$

for $x, 0 < |x-1| < \delta$.

Or, we could use Sequential Criterion. Let $\lim_{n\to\infty} x_n = 1$. By Limit Theorem $\lim_{n\to\infty}(x_n^2 - x_n + 1) = 1$ and $\lim_{n\to\infty}(x_n+1) = 2$. Therefore,

$$\lim_{n\to\infty}\frac{x_n^2-x_n+1}{x_n+1} = \frac{\lim_{n\to\infty}(x_n^2-x_n+1)}{\lim_{x\to\infty}(x_n+1)} = \frac{1}{2} \ .$$

(12d). We claim that $\lim_{x\to 0}\sin(1/x^2)$ does not exist. Take the sequence $x_n = \sqrt{1/(2n\pi)}$ and $y_n = \sqrt{1/(2n\pi+\pi/2)}, n \ge 1$. Both sequences tend to 0 as $n \to \infty$. As $\lim_{n\to\infty}\sin(1/x_n^2) = 0$ and $\lim_{n\to\infty}\sin(1/y_n^2) = 1$, they have different limit. We conclude that the limit of $\sin(1/x^2)$ as $x \to 0$ does not exist.

(15). (a) We want to show $\lim_{x\to 0} f(x) = 0$ where f is the function that is equal to x at rational x and 0 at irrational x. The desired conclusion follows from the observation $|f(x)| \le |x|$ and $\lim_{x\to 0}|x| = 0$ and the Squeeze Theorem.

(b) f has no limit at $x = c \ne 0$. Let $x_n \to c$ be a sequence of rational numbers. Clearly, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$. But, take $y_n \to c$ be a sequence of irrational numbers, then $f(y_n) = 0$, so $\lim_{n\to\infty} f(y_n) = 0$. From Sequential Criterion we draw the desired conclusion.

Section 4.2 no. 1bc, 11 cd, 12.

(1b). Since the limit is taken among positive x only, this should be viewed as a right limit (see below). By Limit Theorem,

$$\lim_{x\to 1^+}\frac{x^2+2}{x^2-2} = \frac{\lim_{x\to 1^+}(x^2+2)}{\lim_{x\to 1^+}(x^2-2)} = \frac{3}{-1} = -3.$$

(11c). Let $x_n = 1/(2n\pi) \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \operatorname{sgn}\sin 1/x_n = \lim_{n\to\infty} \operatorname{sgn}0 = 0 \ .$$

On the other hand, let $y_n = 1/(2n\pi+\pi/2) \to 0$ as $n \to \infty$. We have

$$\lim_{n\to\infty} \operatorname{sgn}\sin 1/y_n = \lim_{n\to\infty} \operatorname{sgn}\ 1 = 1.$$

We conclude from Sequential Criterion that the limit does not exist.

(11d). Using the inequality

$$|\sqrt{x}\sin 1/x^2| \le \sqrt{x} \ ,$$

and the fact that $\lim_{x\to 0^+}\sqrt{x}=0$, we conclude from Squeeze Theorem that

$$\lim_{x\to 0^+}\sqrt{x}\sin\frac{1}{x^2}=0\ .$$

Supplementary Exercises

1. Let f be function defined on (a,b) except possibly at $x_0\in(a,b)$. It is has a **right hand limit** at x_0 if there exists some L such that for all $\varepsilon>0$, there exists some $\delta>0$ such that $|f(x)-L|<\varepsilon$ for all $x\in(x_0,x_0+\delta)\cap(a,b)$. Denote it by $L=\lim_{x\to x_0^+}f(x)$. Similarly we define the **left hand limit** of f at x_0 and denote it by $\lim_{x\to x_0^-}f(x)$. Show that $\lim_{x\to x_0}f(x)$ exists if and only if both one-sided limits exist and are equal.

   Solution. $\Rightarrow$. When $\lim_{x\to x_0}f(x)=L$, for $\varepsilon>0$, there is some δ such that $|f(x)-L|<\varepsilon$ for $0<|x-x_0|<\delta, x\in(a,b)$. Certainly it means $|f(x)-L|<\varepsilon$ for $x\in(x_0,x_0+\delta), x\in(x_0,b)$, and $x\in(x_0-\delta,x_0), x\in(a,x_0)$. In other words, both one-sided limits exist and equal.

   $\Rightarrow$. Let $L=\lim_{x\to x_0^+}f(x)=\lim_{x\to x^-}f(x)$. For $\varepsilon>0$, there exists δ_1 such that $|f(x)-L|<\varepsilon$ for $x\in(x_0,x_0+\delta_1), x\in(x_0,b)$. On the other hand, there exists δ_2 such that $|f(x)-L|<\varepsilon$ for $x\in(x_0-\delta_2), x\in(a,x_0)$. Therefore, by taking $\delta=\min\{\delta_1,\delta_2\}$, $|f(x)-L|<\varepsilon$ for all $x\in(a,b), 0<|x-x_0|<\delta$, that is, $\lim_{x\to x_0}f(x)=L$.

2. Let f be defined on (a,b) possibly except $x_0\in(a,b)$. Show that $\lim_{x\to x_0}|f(x)|=|L|$ whenever $\lim_{x\to x_0}f(x)=L$.

   Solution. It follows immediately from the triangle inequality $||f(x)|-|L||\le|f(x)-L|$.

3. Let f be defined on (a,b) possibly except $x_0\in(a,b)$. Suppose that $\lim_{x\to x_0}f(x)=L$ for some L. Show that $\lim_{x\to x_0}\sqrt{f(x)}=\sqrt{L}$ provided $f\ge 0$ on (a,b). Suggestion: Consider $L>0$ and $L=0$ separately.

   Solution. First, assume $L>0$. Given $\varepsilon=L/2>0$, there is some δ_1 such that $|f(x)-L|\le L/2$ for $0<|x-x_0|<\delta_1$. In particular, it implies that $f(x)\ge L/2$ for $0<|x-x_0|<\delta_1$. Now,
   $$|\sqrt{f(x)}-L^{1/2}|=\frac{|f(x)-L|}{\sqrt{f(x)}+L^{1/2}}\le\frac{1}{(L/2)^{1/2}+L^{1/2}}\times|f(x)-L|\ ,$$
   for $0<|x-x_0|<\delta_1$. For $\varepsilon>0$, there is δ_2 such that $|f(x)-L|<\varepsilon\times[(L/2)^{1/2}+L^{1/2}]$ for $x, 0<|x-x_0|<\delta_2$. If we take $\delta=\min\{\delta_1,\delta_2\}$, then
   $$|\sqrt{f(x)}-L^{1/2}|<\frac{1}{(L/2)^{1/2}+L^{1/2}}\times|f(x)-L|<\varepsilon\ ,\quad \forall x, 0<|x-x_0|<\delta,$$
   done.

   Next, $L=0$. Given $\varepsilon>0$, there is some δ such that $|f(x)|<\varepsilon^2$ for all $x, 0<|x-x_0|<\delta$. It follows that $|\sqrt{f(x)}-0|=\sqrt{f(x)}<\varepsilon$ for $x, 0<|x-x_0|<\delta$, done.